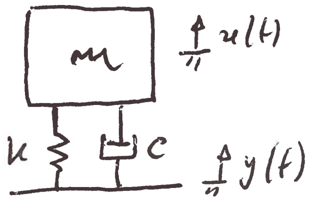
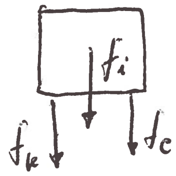
Lodz University of Technology

VR_2 2019/20

P1

a)





Assumed $u > y$

$$f_i = m\ddot{u}$$
$$f_k = k(u-y)$$
$$f_c = c(\dot{u}-\dot{y})$$
$$m\ddot{u} + c(\dot{u}-\dot{y}) + k(u-y) = 0$$
$$m\ddot{u} + c\dot{u} + ku = c\dot{y} + ky$$

b) $T = \frac{1}{2} m \dot{u}^2$ $\quad V = \frac{1}{2} k (u-y)^2$ $\quad \mathcal{F} = \frac{1}{2} c (\dot{u}-\dot{y})^2$

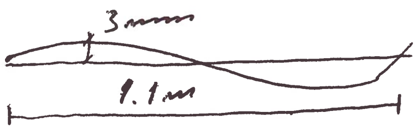
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \underbrace{\cancel{\frac{\partial T}{\partial u}}}_{=0} + \frac{\partial V}{\partial u} + \frac{\partial \mathcal{F}}{\partial \dot{u}} = \underbrace{\cancel{Q}}_{=0}$$

$$m\ddot{u} + k(u-y) + c(\dot{u}-\dot{y}) = 0$$
$$m\ddot{u} + c\dot{u} + ku = c\dot{y} + ky$$

c) $$TR = \sqrt{\frac{k^2 + (\omega c)^2}{(k-\omega^2 m)^2 + (\omega c)^2}} = \sqrt{\frac{1 + (2\zeta\beta)^2}{(1-\beta^2)^2 + (2\zeta\beta)^2}}$$

$m = 500\,kg$ $\quad k_{eq} = 5.25 \times 10^6\,N/m$ $\quad c_{eq} = 20 \times 10^3\,Ns/m$

$v = 61.2\,km/h = 17\,m/s$ $\quad v = \frac{L}{T}$ $\quad T = \frac{L}{v} = \frac{1.1}{17} = 0.0647\,s$



$T = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{0.0647} = 97.1\,rad/s$

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$$TR = \sqrt{\frac{(5.25\times10^6)^2 + (97.1\times 20\times10^3)^2}{(5.25\times10^6 - 97.1^2\times500)^2 + (97.1\times20\times10^3)^2}} = \underline{2.78}$$

A amplitude máxima de resposta é 2.78 vezes superior à amplitude imposta.

d) A amplitude de resposta será $2.78\times3\times10^{-3} = 8.34\times10^{-3}$ m

falta a desfasagem.

$$m\ddot{u} + c\dot{u} + ku = c\dot{y} + ky$$

$$y = Y \operatorname{sen} \omega t = 3\times10^{-3} \operatorname{sen} 97.1t$$

$$\dot{y} = \omega Y \cos \omega t = 97.1\times3\times10^{-3} \cos 97.1t = 0.2913 \cos 97.1t$$

$$500\ddot{u} + 20\times10^3\dot{u} + 5.25\times10^6 u = 20\times10^3\times0.2913 \cos 97.1t + 5.25\times10^6\times3\times10^{-3} \operatorname{sen} 97.1t$$

$$500\ddot{u} + 20\times10^3\dot{u} + 5.25\times10^6 u = \underbrace{5826 \cos 97.1t + 15750 \operatorname{sen} 97.1t}_{} = F\operatorname{sen}(97.1t+\varphi) = F\operatorname{sen}97.1t\cos\varphi + F\cos97.1t\operatorname{sen}\varphi$$

$$5826 = F\operatorname{sen}\varphi$$
$$15750 = F\cos\varphi$$
$$\Rightarrow F = \sqrt{5826^2 + 15750^2} = 16793\ N$$
$$\operatorname{tg}\varphi = 0.3699 \Rightarrow \varphi = 0.354\ \text{rad}$$

$\therefore$

$$500\ddot{u} + 20\times10^3\dot{u} + 5.25\times10^6 u = 16793 \operatorname{sen}(97.1t + 0.354)$$

para $F \operatorname{sen}\omega t$ a resposta é $X \operatorname{sen}(\omega t - \alpha)$, c/ $\alpha = \operatorname{tg}^{-1}\frac{\omega c}{k-\omega^2 m}$

$$\alpha = \operatorname{tg}^{-1}\frac{97.1\times20\times10^3}{5.25\times10^6 - 97.1^2\times500} = 1.3016\ \text{rad}$$

mas $f(t)$ já tem uma desfasagem à partida, para $F\operatorname{sen}(\omega t+\varphi)$ a resposta será $X\operatorname{sen}(\omega t + \varphi - \alpha)$, i.e,

$$u(t) = 8.34\times10^{-3} \operatorname{sen}(97.1t + 0.354 - 1.3016)$$

$$u(t) = 8.34\times10^{-3} \operatorname{sen}(97.1t - 0.9476)$$

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A amplitude também se pode calcular a partir de

$$X = \frac{F}{\sqrt{(k-\omega^2 m)^2 + (\omega c)^2}} = \frac{16793}{\sqrt{(5.25\times10^6 - 97.1^2\times 500)^2 + (97.1\times 20\times10^3)^2}}$$

$$= 8.34\times10^{-3}\ \mathrm{m}$$

e) Força transmitida pela estrada: é através de k e c:

$$f_{tr}(t) = c(\dot{u}-\dot{y}) + k(u-y)$$

Da eq. de equilíbrio, $m\ddot{u} + c(\dot{u}-\dot{y}) + k(u-y) = 0$

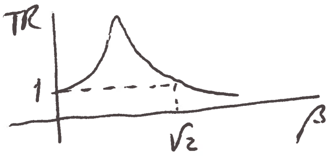
Donde $f_{tr}(t) = -m\ddot{u}(t) = +\omega^2 m\, u(t) =$

$$= 97.1^2 \times 500 \times 8.34\times10^{-3}\ \mathrm{sen}\,(97.1t - 0.9476) =$$

$$= 39316.5\ \mathrm{sen}\,(97.1t - 0.9476)$$

O valor máximo é $F_{tr} = 39316.5\ \mathrm{N}$

f) $X \leq Y \Rightarrow \frac{X}{Y} \leq 1 \Rightarrow TR \leq 1 \Rightarrow \beta \geq \sqrt{2}$, qualquer que seja o amortecimento.



$\frac{\omega}{\omega_n} \geq \sqrt{2}$ $\quad \omega \geq \sqrt{2}\,\omega_n$ $\quad$ Como $v = \frac{L}{T} = \frac{L\omega}{2\pi}$, $\omega = \frac{2\pi}{L} v$

$\frac{2\pi}{L} v \geq \sqrt{2}\,\omega_n$ $\quad v \geq \frac{L}{2\pi}\sqrt{2}\,\omega_n$ $\quad v \geq \frac{1.1}{2\pi}\sqrt{2}\cdot\sqrt{\frac{5.25\times10^6}{500}}$

$v \geq 25.37\ \mathrm{m/s}$

$v \geq 91.3\ \mathrm{km/h}$

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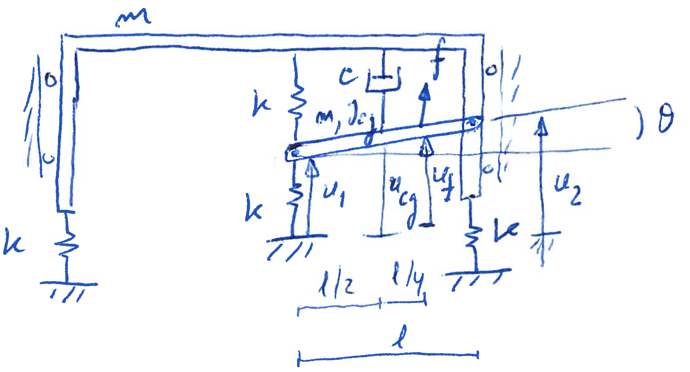
P2



a) i) $T = \frac{1}{2} m \dot{u}_2^2 + \frac{1}{2} m \dot{u}_{cg}^2 + \frac{1}{2} J_{cg}\dot{\theta}^2$

$u_{cg} = \frac{u_1+u_2}{2}$

$\theta = \frac{u_2-u_1}{l}$

$J_{cg} = \frac{1}{12} m l^2$

$$T = \frac12 m\dot u_2^2 + \frac12 m\left(\frac{\dot u_1+\dot u_2}{2}\right)^2 + \frac12\frac1{12} m \cancel{l^2}\frac{(\dot u_2-\dot u_1)^2}{\cancel{l^2}}$$

$$T = \frac12 m \dot u_2^2 + \frac12 \frac m4 (\dot u_1+\dot u_2)^2 + \frac12\frac1{12} m(\dot u_2-\dot u_1)^2$$

ii) $V = \frac12 2k u_2^2 + \frac12 k u_1^2 + \frac12 k(u_2-u_1)^2$

iii) $\mathcal{F} = \frac12 c(\dot u_2 - \dot u_{cg})^2 = \frac12\frac c4(\dot u_2-\dot u_1)^2$

$u_2 - u_{cg} = u_2 - \frac{u_1+u_2}{2} = \frac{u_2-u_1}{2}$

iv) $\delta W = f\,\delta u_f = Q_1\delta u_1 + Q_2 \delta u_2$

$\delta W = \frac f4 \delta(u_1+3u_2) = Q_1\delta u_1 + Q_2\delta u_2$

$u_f = \frac{u_{cg}+u_2}{2} = \frac{\frac{u_1+u_2}{2}+u_2}{2} = \frac{u_1}{4}+\frac{u_2}{4}+\frac{u_2}{2} = \frac{u_1}{4}+\frac34 u_2 = \frac14(u_1+3u_2)$

$\frac f4\delta u_1 + \frac34 f\delta u_2 = Q_1\delta u_1+Q_2\delta u_2$

$Q_1 = f/4 \;,\; Q_2 = 3f/4$

b) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot u_i}\right) - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} + \frac{\partial \mathcal{F}}{\partial \dot u_i} = Q_i \quad i=1,2$

$u_1: \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u_1}\right) = \frac m4(\ddot u_1+\ddot u_2) + \frac1{12} m(\ddot u_2-\ddot u_1)(-1) =$

$= \frac m3\ddot u_1 + \frac m6 \ddot u_2$

$\frac{\partial T}{\partial u_1} = 0$

$\frac{\partial V}{\partial u_1} = k u_1 + k(u_2-u_1)(-1) = 2ku_1 - ku_2$

$\frac{\partial \mathcal{F}}{\partial \dot u_1} = \frac c4(\dot u_2-\dot u_1)(-1) = \frac c4(\dot u_1-\dot u_2)$

$Q_1 = f/4$

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$\therefore$

$$\frac{m}{3}\ddot{u}_1 + \frac{m}{6}\ddot{u}_2 + \frac{c}{4}\dot{u}_1 - \frac{c}{4}\dot{u}_2 + 2ku_1 - ku_2 = f/4$$

u_2 :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}_2}\right) = m\ddot{u}_2 + \frac{m}{4}(\ddot{u}_1+\ddot{u}_2) + \frac{m}{12}(\ddot{u}_2-\ddot{u}_1), \quad \frac{\partial T}{\partial u_2} = 0$$

$$= \frac{m}{6}\ddot{u}_1 + \frac{4}{3}m\ddot{u}_2$$

$$\frac{\partial V}{\partial u_2} = 2ku_2 + k(u_2-u_1) = -ku_1 + 3ku_2$$

$$\frac{\partial \mathcal{F}}{\partial \dot{u}_2} = \frac{c}{4}(\dot{u}_2-\dot{u}_1) = -\frac{c}{4}\dot{u}_1 + \frac{c}{4}\dot{u}_2$$

$$Q_2 = 3f/4$$

$\therefore$

$$\frac{m}{6}\ddot{u}_1 + \frac{4}{3}m\ddot{u}_2 - \frac{c}{4}\dot{u}_1 + \frac{c}{4}\dot{u}_2 - ku_1 + 3ku_2 = 3f/4$$

$$\begin{bmatrix} \frac{m}{3} & \frac{m}{6} \\ \frac{m}{6} & \frac{4}{3}m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} \frac{c}{4} & -\frac{c}{4} \\ -\frac{c}{4} & \frac{c}{4} \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f/4 \\ 3f/4 \end{Bmatrix}$$

c)

$$\begin{bmatrix} 2k - \omega^2\frac{m}{3} & -k-\omega^2\frac{m}{6} \\ -k-\omega^2\frac{m}{6} & 3k-\omega^2\frac{4}{3}m \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\left(2k-\omega^2\frac{m}{3}\right)\left(3k-\omega^2\frac{4}{3}m\right) - \left(-k-\omega^2\frac{m}{6}\right)^2 = 0$$

$$6k^2 - \omega^2 km - \omega^2\frac{8}{3}km + \omega^4\frac{4}{9}m^2 - k^2 - \omega^4\frac{m^2}{36} - \omega^2\frac{1}{3}km = 0$$

$$\frac{15}{36}m^2\omega^4 - 4km\omega^2 + 5k^2 = 0$$

$$\frac{5}{12}m^2\omega^4 - 4km\omega^2 + 5k^2 = 0 \qquad \omega^2 = \frac{4km \mp \sqrt{16k^2m^2 - 4\,\frac{25}{12}k^2m^2}}{\frac{5}{6}m^2}$$

$$\omega^2 = \frac{4 \mp \sqrt{16-\frac{25}{3}}}{5/6}\,\frac{k}{m} \qquad k = 10^4\ N/m$$

$$m = 1kg$$

$$\omega_1^2 = 14773.5\ s^{-2} \rightarrow \omega_1 = 121.55\ rad/s$$

$$\omega_2^2 = 81226.5\ s^{-2} \rightarrow \omega_2 = 285\ rad/s$$

1.º modo:

$$\left(2k - \omega_1^2 \frac{m}{3}\right) u_1^{(1)} - \left(k + \omega_1^2 \frac{m}{6}\right) u_2^{(1)} = 0$$

$$\left(2\times10^4 - 14773.5 \times \frac{1}{3}\right) u_1^{(1)} - \left(10^4 + 14773.5 \times \frac{1}{6}\right) u_2^{(1)} = 0$$

$$u_1^{(1)} = 1 \Rightarrow u_2^{(1)} = 1.21 \quad \rightarrow \quad \begin{Bmatrix} 1 \\ 1.21 \end{Bmatrix}$$

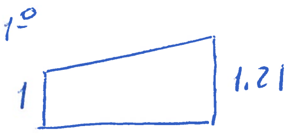
2.º modo:

$$\left(2k - \omega_2^2 \frac{m}{3}\right) u_1^{(2)} - \left(k + \omega_2^2 \frac{m}{6}\right) u_2^{(2)} = 0$$

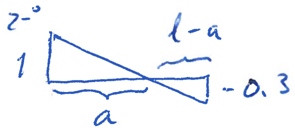
$$\left(2\times10^4 - 81226.5 \times \frac{1}{3}\right) u_1^{(2)} - \left(10^4 + 81226.5 \times \frac{1}{6}\right) u_2^{(2)} = 0$$

$$u_1^{(2)} = 1 \Rightarrow u_2^{(2)} = -0.3 \quad \rightarrow \quad \begin{Bmatrix} 1 \\ -0.3 \end{Bmatrix}$$

d)



Quando u_1 anda 1, u_2 anda 1.21 (no mesmo sentido)



Quando u_1 anda 1, u_2 anda -0.3 (no sentido oposto)

Nodo: $\dfrac{1}{a} = \dfrac{0.3}{l-a} \Rightarrow l - a = 0.3a \Rightarrow l = 1.3a \Rightarrow a = \dfrac{l}{1.3} \simeq 0.15\,m$

O nodo está a aprox. 0.15 m de u_1.

f)

$$\begin{bmatrix} 2\times10^4 - 150^2 \times \frac{1}{3} & -10^4 - 150^2 \times \frac{1}{6} \\ -10^4 - 150^2 \times \frac{1}{6} & 3\times10^4 - 150^2 \times \frac{4}{3} \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 100/4 \\ 3\times100/4 \end{Bmatrix}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{matrix} 5.45 \times 10^{-3}\,m \\ 6.78 \times 10^{-3}\,m \end{matrix} \qquad \begin{cases} u_1(t) = 5.45 \times 10^{-3} \operatorname{sen} 150t \\ u_2(t) = 6.78 \times 10^{-3} \operatorname{sen} 150t \end{cases}$$

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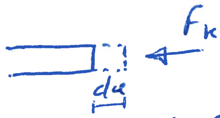
P3

a) $AE\,u'(l,t) = -k\,u(l,t)$

A condição de fronteira em $x=l$ é: quando a barra sofre um elongamento positivo, a mola exerce uma força no sentido negativo.



A força na face $x=l$ é $P=\sigma A$. A força da mola, F_K é $-k\,u(l,t)$. Portanto,

$$\sigma A = -k\,u(l,t)$$

Como $\sigma = E\varepsilon = E\dfrac{\partial u}{\partial x} = E u'$,

$$AE \left.\frac{\partial u}{\partial x}\right|_{x=l} = -k\,u(l,t)$$

$$AE\,u'(l,t) = -k\,u(l,t)$$

Como $u(l,t) = \phi(l)\,T(t)$,

$$AE\,\phi'(l)\,T(t) = -k\,\phi(l)\,T(t)$$

Para todo o tempo, $AE\,\phi'(l) = -k\,\phi(l)$

b) Cond. de fronteira: $x=0 \Rightarrow u(0,t)=0 \Rightarrow \phi(0)=0$

$x=l \Rightarrow AE\,u'(l,t) = -k\,u(l,t)$

$\Rightarrow AE\,\phi'(l) = -k\,\phi(l)$

Como $\phi(x) = \underset{\sim}{A}\cos\frac{\omega}{c}x + \underset{\sim}{B}\,\text{sen}\,\frac{\omega}{c}x$, c/ $c=\sqrt{\frac{E}{\rho}}$,

da 1ª condição vem $0 = \underset{\sim}{A} + 0 \Rightarrow \underset{\sim}{A} = 0$

$$\phi'(x) = \underset{\sim}{B}\,\frac{\omega}{c}\cos\frac{\omega}{c}x$$

Da 2ª condição,

$$AE\,\underset{\sim}{B}\,\frac{\omega}{c}\cos\frac{\omega}{c}l = -k\,\underset{\sim}{B}\,\text{sen}\,\frac{\omega}{c}l$$

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$$AE\,\frac{\omega}{c}\cos\frac{\omega}{c}l = -k\,\mathrm{sen}\frac{\omega}{c}l$$

$$-\frac{AE}{k}\,\frac{\omega}{c} = \mathrm{tg}\,\frac{\omega}{c}l \quad , \text{ou} \quad -\frac{AE}{kl}\,\frac{\omega}{c}l = \mathrm{tg}\,\frac{\omega}{c}l$$

Fazendo $\alpha = \frac{\omega}{c}l$, $\quad -\frac{AE}{kl}\alpha = \mathrm{tg}\,\alpha$

$$\therefore \quad \boxed{\frac{\mathrm{tg}\,\alpha}{\alpha} = -\frac{AE}{kl}} \quad \text{eq. de frequências}$$

c) Da eq. de frequências, sai $\alpha_n = \frac{\omega_n}{c}l$, $n = 1, \dots \infty$

Portanto, $\omega_n = \frac{\alpha_n c}{l}$, $n = 1, \dots \infty$

Modos: $\phi^{(n)}(x) = \underset{\sim}{B}_n\,\mathrm{sen}\,\frac{\omega_n}{c}x$

$\phi^{(n)}(x) = \underset{\sim}{B}_n\,\mathrm{sen}\,\frac{\alpha_n}{l}x$, $n = 1, \dots \infty$

d) $u^{(n)}(x,t) = \phi^{(n)}(x)\,T_n(t) = \underset{\sim}{B}_n\,\mathrm{sen}\,\frac{\alpha_n}{l}x\,(C_n\cos\omega_n t + D_n\,\mathrm{sen}\,\omega_n t)$

Esta é a resposta de cada modo.

A resposta total é a sobreposição das respostas de todos os modos:

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{sen}\,\frac{\alpha_n}{l}x\,(C_n\cos\omega_n t + D_n\,\mathrm{sen}\,\omega_n t)$$

em que C_n e D_n são calculados aplicando as condições iniciais de deslocamento e velocidade, $u(x,0)$, $\dot{u}(x,0)$, i.e., $u_0(x)$ e $\dot{u}_0(x)$.

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P4

a)
$$L_{p_1} = 10\lg 5 + 120 - 10\lg(2\pi \times 10^2) - 11 = 88\,dB(A)$$
$$L_{p_2} = 10\lg 2 + 120 - 10\lg(2\pi \times 10^2) - 5 = 90\,dB(A)$$
$$\Sigma E_i = 100 = \frac{\Delta t}{40} 10^{(0.1\times(88-67))} + \frac{\Delta t}{40} 10^{0.1\times(90-67)}$$
$$\Delta t = 12.3\,h$$

b)
$$L_{p_1} = 116 - 10\lg(2\pi r^2)$$
$$L_{p_2} = 118 - 10\lg(2\pi r^2)$$
$$L_{p_T} = 87 = 10\lg\left(10^{11.6 - \lg 2\pi r^2} + 10^{11.8 - \lg 2\pi r^2}\right)$$
$$10^{8.7} = \frac{10^{11.6} + 10^{11.8}}{2\pi r^2} \qquad r = 18.1\,m$$

c)
$$L_{p_1} = 86\,dB(A)$$
$$L_{p_2} = 85\,dB(A) \Rightarrow 100 = \frac{\Delta t}{40} 10^{0.1(86-67)} + \frac{\Delta t}{40} 10^{0.1(85-67)}$$
$$\Delta t = 28.1\,h$$

Consegue-se aumentar o tempo cerca de 2,3 vezes.

Your success zone

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